PHOTOELASTICITY

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Key Words: photoelasticity, stress, strain, experimental stress analysis, birefringence

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Summary

Photomechanics encompasses experimental techniques that use properties of light propagating through loaded or deformed components to determine and analyze the relative displacements in the material in order to establish their strain and stress fields. Photoelasticity is a branch of Photomechanics. It employs models constructed from materials transparent to the used light that exhibit birefringence under applied stress and are observed under polarized light in an instrument called polariscope. The photoelastic response consists of two families of fringes that are observed in the polariscope: isochromatic and isoclinic fringes.

Photoelasticity may be applied to models in the laboratory or to prototypes in the field and to 2D or to 3D studies. It is a whole field technique. While the optical response shows stress distributions over relatively large spatial dimensions, it allows accurate determination of stress states in localized areas or points of a component. As a consequence, photoelasticity indicates not only the most loaded areas of the observed component, but also can provide accurate stress values at any critical points.
1. Introduction

The chapter Photomechanics encompasses experimental techniques that use properties of light propagating through loaded or deformed components to determine and analyze the relative displacements in the material in order to establish their strain and stress fields.

Photoelasticity is a branch of Photomechanics. It employs models constructed from materials transparent to the light that exhibit birefringence under applied stress and are observed under polarized light in an instrument called polariscope.

Photoelasticity may be applied to models in the laboratory or to prototypes in the field. It is a whole field technique. While the optical response shows stress distributions over relatively large spatial dimensions, it allows accurate determination of stress states in localized areas or points of a component. As a consequence, photoelasticity indicates not only the most loaded areas of the observed component, but also can provide accurate stress values at any critical points. Photoelasticity may be applied to 2D or to 3D studies and can be extended to non-linear elastic, elastic-plastic and dynamic problems. In these cases the techniques are most appropriately called Non-linear Photomechanics, Photoplasticity and Photodynamics.

Some examples of photoelasticity are shown here. Figure 1a shows the localized photoelastic response used to determine the stress concentration factor of a sharp U notch. Figure 2b shows the whole field response of a C shaped model loaded by compressive forces. Qualitative observation of the optical response allows the identification of most stressed areas that may be inspected quantitatively under higher magnification or by another experimental technique. Figure 1c shows residual fringes caused by the fabrication process of bulbs for incandescent lamps. In this case photoelasticity is used as an inspection tool in the production of glass hardware.

The photoelastic response consists of two families of characteristic lines that are observed in the polariscope: isochromatic and isoclinic fringes. The isochromatic fringes correspond to the geometric locus of material points that present the same principal stress differences. The stress optic law (1) relates the principal stress differences $\sigma_i - \sigma_j$ with the measured isochromatic fringe order N where t is the thickness of the model at the point under analysis and $f_\sigma$ is the stress fringe value that depends on the photoelastic material and the wave length of light used in the observation.

![Figure 1](image1.png)

Figure 1: Examples of photoelasticity. (a) stress concentration area of a sharp U-notch, (b) whole field stress distribution image of a C-shaped model, (c) residual stresses in lamp bulbs caused by the fabrication process.
\[ \sigma_I - \sigma_{II} = \frac{N}{t} f_\sigma \]  

The isoclinic fringes correspond to the geometric locations of the observed material points which make 0° or 90° with the polariscope axes.

The polariscope is the basic instrument of the photoelastic experiment. It basically consists of a light source, two plates of linear polarizers and two plates of wave retarders. Besides other possible arrangements, the polarisopes are generally employed in one of the two configurations: plane polariscope - uses linear or plane polarized light and shows the families of isochromatic and isoclinic fringes; circular polariscope – uses circularly polarized light to show the family of isochromatic fringes. A sketch of a polariscope which includes a loaded model is presented in Figure 2.

Figure 2: A loaded model inserted in the working field of a transmission light polariscope that consists of a light source, two linear polarizers, two wave retarders and an observer.

Usually the plane polariscope uses the linear polarizers in a crossed arrangement. A common arrangement of the circular polariscope also uses crossed wave retarders. The polariscope depicted in the Figure 2 works in a plane polariscope mode since the principal axes of wave retarders were placed parallel to the axes of the linear polarizers in order to be ineffective.

2. Light
The electromagnetic theory of light propagation is used to adequately explain the photoelastic effect. The wave equation is presented in equation (2). The solution of the wave equation is the space of harmonic functions. These functions can be represented by a series combination of sine functions with arguments given by multiples of the basic frequency \( w \) (rd/s) or \( f \) (Hertz).

If white light is used all visible wave lengths \( \lambda \) will compose the harmonic function. If only one wave length is used (monochromatic light) the solution of the unidirectional wave equation (3) with light propagating in the positive direction is given by equation (4). In this last equation \( E \) is the propagating light vector which is parallel to a given plane and its amplitude is given by \( E_0 \). Vector \( E \) can be seen as the sum of two orthogonal vectors that propagate in the planes \( x \) and \( y \).

\[
\nabla^2 E = \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} \tag{2}
\]

\[
\frac{\partial^2 E}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} \tag{3}
\]

\[
E = E_0 \cos(\omega t + \phi) \quad w = 2\pi f \quad T = \frac{1}{f} \quad \lambda = T \cdot c \tag{4}
\]

3. Linear Polarizers

Light generated at a diffuse light source propagates along all possible directions. For each direction the electrical vectors have components in all possible planes. Figure 3 shows light propagating along a specific direction and passing through a polarizer filter. After crossing the polarizer the light wave will be represented by the components of the electrical vector that propagates parallel to the plane of polarization of the filter.

Figure 3: Propagation of light across a plane polarizer filter
When light propagates parallel to a given plane it is called plane or linearly polarized. Light may be made linearly polarized by reflection from a flat surface if an adequate incidence angle is chosen for the incident light ray. Equation (5) and Figure 4 depict the $E_{0}^{\parallel}$ (parallel) $E_{90}^{\perp}$ (orthogonal) components of the incident and reflected light rays that hit a reflective surface. Equation (5) shows that the reflected and incident light intensities $I_r$ and $I_i$ are related by the reflection coefficient $R$, which is different for the parallel and orthogonal waves. For the air-glass media interface there is a critical angle $\alpha_p = 57^\circ$ such that the reflected component $E_{ry}$ is zero. Figure 4 illustrates this phenomenon.

Another way of generating large field of views of polarized light is accomplished by using initially transparent plastic sheet of polyvinyl alcohol impregnated by iodine. The necessary orientation of the molecules is achieved by stretching. (for example sheets fabricated by Polaroid Inc. as types HN22, HN32 and HN38) such as shown in Figure 5. The filter classified as HN22 is the most convenient to be used in photoelasticity.

![Figure 4: Incident, reflective and refracted waves](image)

\[ I_r = R.I_i \]
\[ R_{\parallel} = \frac{\tan^2(\alpha - \gamma)}{\tan^2(\alpha + \gamma)} \quad R_{90^\circ} = \frac{\sin^2(\alpha - \gamma)}{\sin^2(\alpha + \gamma)} \]  
\[ \sin \alpha = \frac{n_2}{n_1} \quad \sin \gamma \]

\[ n_1 \]
Figure 5: Polarizer filters used in a demonstration of photoelasticity: (a) the overhead projector is used as the light source. Polarizer filters are superposed with axes crossed. A loaded C model is placed between the filters; (b) the laptop screen that emits linearly polarized light is used together with a polarizer filter to show an arrangement of a plane polariscope and isochromatic and isoclinic fringes generated in a loaded C-shaped model.

4. Wave Plate

The wave plate transmits the components of the light through two orthogonal planes called F (fast) and S (slow). The components $E_F$ and $E_S$ of the light vector $E$ cross the filter thickness with different velocities causing a relative retardation between $E_F$ and $E_S$ and consequently an elliptic propagation of the light after emerging from the filter. This behavior is depicted in Figure 6, where the relative linear retardation $\delta$ (or angular retardation $\Delta = 2 \pi \delta / \lambda$) is schematically shown between the components $E_F$ and $E_S$.

Figure 6: Light crossing a wave plate where F and S are respectively the fast and the slow directions of propagation of the electrical light vector.
Equations (6a) and (6b) respectively establish the conditions for the resulting elliptic light propagate as a circular or a linear polarized light.

Circular polarized light if: \[ \begin{aligned}
\delta &= \frac{\lambda}{4} \\
E_F &= E_S
\end{aligned} \]  \hspace{1cm} (6a)

Linear or plane polarized light if: \( \delta = 0 \) \hspace{1cm} (6b)

The wave plate with a phase shift of \( \pi/2 \) is often used in photoelasticity and is called quarter-wave plate.

5. Plane Polariscope

The plane polariscope consists of a light source and two linear polarizer plates that usually are employed with their polarization axes crossed (Figure 7). The light intensity I sensed by the observer is zero if there is no stressed model in the working field. The plane polariscope is used for the observation of isochromatic and isoclinic fringes.

![Figure 7: Plane polariscope with crossed polarizer plates.](image)

6. Circular Polariscope

The circular polariscope is used for the observation of isochromatic fringe orders. The name ‘circular’ comes from the fact that it uses circular propagating light in its working field (Figure 8). The light wave is generated in the light source and crosses the polarizer plate to be plane polarized. A \( \frac{1}{4} \) wave plate is placed with its fast and slow axes making a 45° angle with the polarizer. This angle enables both amplitudes \( E_S \) and \( E_F \) become equal. The second wave plate is arranged to be crossed to the first one. This arrangement regenerates the initial plane of polarization. The observer does not sense light intensity in this arrangement unless a loaded birefringent model is positioned in the working field.
zero light intensity is caused by the placement of an analyzer with axes crossed to the first polarizer. Other arrangements of axes of polarizers and wave plates are possible and are used in some applications.

![Diagram of circular polariscope in a crossed-crosed arrangement](image)

**Figure 8: Circular polariscope in a crossed-crosed arrangement**

7. Model of Birefringent Material

The photoelastic models are built from birefringent materials. These materials behave as ordinary general wave plates. They cause retardation between the components of the light vector that pass through the deformed material. The relationship between the incident light vector \( \mathbf{E} \) and the state of stress in the observed point of a birefringent material is illustrated in Figure 9.

![Diagram of birefringent material stress effect](image)

**Figure 9: Effect of the state of stress at a point in a loaded model of birefringent material on the incident light vector \( \mathbf{E} \).**
The stress state causes the retardation $\delta$ between the components of the electrical light vector that are parallel to the principal stress directions (defined by an angle $\alpha$).

The relative retardation $\delta$ caused by the stress state depends on the principal stress difference $\sigma_I - \sigma_{II}$, the thickness of the component being observed $t$, the wavelength $\lambda$ of the incident light and the birefringence constant of the material, $K$ (equation 7). The stress fringe value $f_\sigma$ and the strain fringe value $f_\varepsilon$ are also used and their relationship is defined by the material elastic constants $\nu$, the Young modulus $E$ and the Poisson coefficient $\mu$ (Equation 8).

\[
\sigma_I - \sigma_{II} = \frac{\delta}{t} K = \frac{\delta}{t,\lambda} f_\sigma = \frac{N}{t} f_\sigma = \frac{\Delta}{2\pi t} f_\sigma 
\]  
(7)

\[
\varepsilon_I - \varepsilon_{II} = \frac{N}{t} f_\varepsilon \\
f_\varepsilon = \frac{1 + \mu}{E} f_\sigma
\]  
(8)

8. Loaded Model in a Plane Polariscop

Figure 10 shows a loaded model (disk under a diametrical compression) in the working field of a plane polariscop. The intensity of light sensed by an observer or image grabber device is proportional to the square of the amplitude of the electrical vector that passes through the analyzer. Light intensity is given by equation 9 (see the development in section 10 - Jones Calculus). The black fringes (zero light intensity) observed in the whole field image of the model are the isochromatic and the isoclinic fringe families.

\[
I \propto E_o^2 \propto E_o^2 \cdot \sin^2 \theta \cdot 2\alpha \cdot \sin^2 \frac{\Delta}{2}
\]  
(9)
Light intensity will be zero when:

\[ I = 0 \quad \text{if} \quad \sin^2 2\alpha = 0 \quad \Leftrightarrow \quad \alpha = 0, \frac{\pi}{2} \quad (10) \]

or

\[ I = 0 \quad \text{if} \quad \sin^2 \frac{\Delta}{2} = 0 \quad \Leftrightarrow \quad \Delta = 0, 2\pi, 4\pi, \ldots \quad (11) \]

The geometric loci of those fringes are the isochromatic and isoclinic fringes. These fringes are shown and numbered in Figure 11. It can be seen that the isoclinic fringes are the locus of all points whose principal stress directions coincide with the polariscope axes, making angles of \( \alpha = 0^\circ \) or \( 90^\circ \). Figure 10 shows material points which principal stress directions make a \( 30^\circ \) angle with the original polariscope axes. In order to observe these points with light intensity zero, the model or the polariscope must be rotated by an angle equal to \( 30^\circ \). The points where light intensities become zero at this position define the \( 30^\circ \) isoclinic.

The isochromatic fringes orders \( N \) are defined by the ratio of relative retardation and wave length, \( \frac{\delta}{\lambda} \) or \( \frac{\Delta}{2\pi} \). Equation (11) shows that whenever retardation corresponds to a full wave length the light intensity becomes zero. Full fringe orders are multiples of the light wave length. They are numbered in Figure 11 for the diametrically compressed disk. Finding full or partial fringe orders and differentiate isoclinics from isochromatic fringes are the job of a photoelastician.

9. Loaded Model in a Circular Polariscope

The use of circular light in the working field makes the light intensity to depend only on the stress difference, \( i.e. \) only isochromatics are observed in the whole field. The light intensity is given by equation (12). Light propagation equations are given in section 11 (Jones Calculus).

Some of the isochromatic fringe values of the diametrically loaded disk are identified in Figure 11. Integer isochromatic fringe \( N=0, 1, 2, 3, \ldots \) orders are generated by the corresponding multiples of full wave lengths retarded, \( i.e., \delta = 0, \lambda, 1\lambda, 2\lambda, \ldots \) or \( \Delta = 0, \pi, 1, 2\pi, \ldots \) as showed in equation (13)

\[ I \propto E_a^2 \propto \sin^2 \frac{\Delta}{2} \quad (12) \]

\[ I = 0 \quad \text{if} \quad \sin^2 \frac{\Delta}{2} = 0 \quad \Leftrightarrow \quad \Delta = 0, 2\pi, 4\pi, \ldots \quad (13) \]
Figure 11: Diametrically compressed disk in a circular polariscope. Isochromatic fringe orders are identified in the photograph from 0 to 9.

10 – Measuring the Partial Fringe Order using the Tardy Compensation Method

The Tardy compensation method is used to determine the fringe order at points that present fractional fringe orders. Light intensities at these points are given by equation (14) and they are not zero in the plane or circular polariscopes as pointed in Figure 12.

\[ I \propto E_a^2 \propto 1 - \cos \Delta \propto \sin^2 \frac{\Delta}{2} \neq 0 \]  

(14)

Figure 12: Measurement of the partial fringe order
The Tardy method makes use of a specific rotation of a specific angle $\gamma$ of the analyzer. The method is explained as follows. At first, it can be shown (see section 11 - Jones Calculus) that a rotation of an angle $\gamma$ of the analyzer causes the light intensity to be given by equation (15).

$$ I \propto E_a^2 \propto 1 - \cos 2\gamma \cos \Delta - \cos 2\alpha \sin 2\gamma \sin \Delta $$

(15)

For this given $\gamma$ rotation of the analyzer a search is conducted to show under which conditions a minimum light intensity $I=0$ occurs. This leads to make the partial derivatives of $I$ with relation to $\alpha$ and $\Delta$ simultaneously equal to zero (equations (16)):

$$ \frac{\partial I}{\partial \alpha} \propto 2 \sin 2\alpha \sin 2\gamma \sin \Delta = 0 $$

$$ \frac{\partial I}{\partial \Delta} \propto \cos 2\gamma \sin \Delta - \cos 2\alpha \sin 2\gamma \cos \Delta = 0 $$

(16)

These equations will be satisfied when

$$ \alpha = \frac{n\pi}{2} \quad e \quad \Delta = 2n\pi \pm 2\gamma $$

(17)

The fringe order at the observed point will be given by

$$ \alpha = \frac{n\pi}{2} \quad e \quad N = \frac{2n\pi \pm 2\gamma}{2\pi} = n \pm \frac{\gamma}{\pi} $$

(18)

The above conditions guarantee that $I=0$ and the fractional fringe order will be given by equation (18). It should be noted that $I$ will only be zero when the principal stress directions at the point are parallel to the polariscope axes.

A procedure commonly used to measure the fractional fringe order follows the steps listed below:

i Select the point at which the fractional fringe order will be measured

ii Set the polariscope at the plane arrangement (deactivate the wave plates by placing their axes parallel to the polarizer and analyzer axes)

iii Rotate the model or the polariscope so that an isoclinic fringe passes over the point, making its light intensity zero and freeze the polariscope at this position

iv Activate the wave plates in their crossed position by rotating them of a 45 angle. This will set the circular polariscope

v Rotate the analyzer of an angle $\gamma$ so that an isochromatic fringe passes over the observed point. Depending on the sense of rotation, the partial fringe will be given by the neighbor fringe plus or minus the rotated angle $\gamma$ divided by $\pi$. Figure 13 helps to clarify this step.
11. Jones Calculus

Polarized light may be mathematically described by a number of methods. The most common are: the Poincare sphere, the Stoke’s vector, the Jones vector, and the quantum mechanical representation. Each of the above methods has advantages and disadvantages. This section discusses the method most widely used in the practical applications of the photoelasticity – the Jones vector approach. It allows describing the polarized beam with simple algebraic expressions and takes care about beams' phase.

An incident electrical vector $E$ has components $E_x$ and $E_y$ such that:

$$E_x = A_x \cdot \cos(wt + \Phi_x) \quad E_y = A_y \cdot \cos(wt + \Phi_y)$$

$$E = \begin{bmatrix} E_x \\ E_y \end{bmatrix} = \text{Re} \left( \begin{bmatrix} A_x e^{i\phi_x} \\ A_y e^{i\phi_y} \end{bmatrix} e^{i\omega t} \right)$$ (19)

where the real Re symbol will be herein omitted.

The mathematical operator that rotates $E_x$ and $E_y$ to give components in the $\theta$ and $\theta + \pi/2$ directions is:

$$R = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$$ (20)

A polarizer allows a light wave to cross its thickness if it propagates along planes parallel to its polarization plane $\beta$. Therefore the mathematical operator of a polarizer $P(\beta)$ is given by:

$$P(\beta) = \begin{bmatrix} \cos \beta & \sin \beta \\ 0 & 0 \end{bmatrix}$$ (21)

For example, polarizer operators that allow light waves passing through planes parallel to the axes $X$, $Y$ and $45^\circ$ are respectively given by the matrices:

$$P(0^\circ) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad P(90^\circ) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad P(45^\circ) = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

Therefore the light wave given in (19) will emerge from the polarizer with components in the $X$ and $Y$ directions given by:
The polarizer operators that allow light waves passing through planes parallel to the axes X, Y and 45° may be referred to the X and Y axes. If the rotation -β is applied the matrices above are respectively given by:

\[
P(0^0)_{X,Y} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad P(90^0)_{X,Y} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad P(45^0)_{X,Y} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}
\]

The birefringent material described in section 7 and illustrated in Figure 9 can be modeled as an ordinary wave plate. The first step to establish the matrix operator to rotate the incident E_x and E_y light vector components. Rotation of the angle θ (by applying the rotation matrix) determines the components of the light vector parallel to the fast F and slow S axes. The mathematical operator that rotates E_x and E_y to give components in the F and S directions is:

\[
R = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}
\]

The component that propagates parallel to the fast axis is advanced by an angle Δ/2 and the component that propagates parallel the slow axis is retarded by an angle Δ/2, so that full relative retardation becomes Δ. The mathematical operator that introduces phase angles in the components of a vector that propagates parallel its F and S directions is given by:

\[
Q = \begin{bmatrix} e^{-i\Delta/2} & 0 \\ 0 & e^{i\Delta/2} \end{bmatrix}
\]

Visualization of the emerging components from the model or wave plate along the X and Y directions requires a rotation of -θ. Therefore, the mathematical operator that models a wave plate or a birefringent model material M with principal directions making an angle θ with the polariscope axes is given by:

\[
M(\theta, \Delta) = W(\theta, \Delta) = \begin{bmatrix} \cos \theta & -\sin \theta & e^{-i\Delta/2} & 0 \\ \sin \theta & \cos \theta & 0 & e^{i\Delta/2} \end{bmatrix} \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}
\]

or

\[
M(\theta, \Delta) = W(\theta, \Delta) = \begin{bmatrix} \cos \Delta/2 - i \sin \Delta/2 \cos \theta & -i \sin \Delta/2 \sin \theta \\ -i \sin \Delta/2 \sin \theta & \cos \Delta/2 + i \sin \Delta/2 \cos \theta \\ -\sin \Delta/2 \cos \theta & \cos \Delta/2 - i \sin \Delta/2 \sin \theta \\ \cos \Delta/2 + i \sin \Delta/2 \sin \theta & -i \sin \Delta/2 \cos \theta \end{bmatrix}
\]

Considering the total relative retardation angle of π/2, the quarter wave plate components of a circular polariscope can be written as:
Quarter wave plate (Fast axis at 45° with X): 
\[ Q_1 = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix} \]

Quarter wave plate (Fast axis at -45° with X): 
\[ Q_2 = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \]

While matrix multiplication is not commutative, it is associative, so a string of multiple matrices representing several devices may be multiplied together yielding a single matrix that describes the optical system as a whole. Therefore it is possible to condense the properties of optical devices acting in series down to a single 2x2 matrix simply by multiplying the matrices of the devices. Examples of strings of multiplication matrices to represent the plane and circular polariscopes and the polariscope with analyzer rotated of an angle \( \gamma \) (Tardy compensation method) are given by equations (26-28):

**Plane polariscope:**

\[
E_{\text{final}} = A(0).M(\Delta, \theta).P(90°) = \\
\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \cos \frac{\Delta}{2} - i \sin \frac{\Delta}{2} \cos 2\theta & -i \sin \frac{\Delta}{2} \sin 2\theta \\ -i \sin \frac{\Delta}{2} \sin 2\theta & \cos \frac{\Delta}{2} + i \sin \frac{\Delta}{2} \cos 2\theta \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A_x.e^{i\phi_x} \\ A_y.e^{i\phi_y} \end{bmatrix} e^{i\omega t} \tag{26}
\]

\[ I \propto \sin^2 2\alpha \sin^2 \frac{\Delta}{2} \tag{9} \]

**Circular polariscope:**

\[
E_{\text{final}} = A(0).Q_2.M(\Delta, \theta).Q_2.P(90°) = \\
\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} i & 1 \\ -i & 1 \end{bmatrix} \begin{bmatrix} \cos \frac{\Delta}{2} - i \sin \frac{\Delta}{2} \cos 2\theta & -i \sin \frac{\Delta}{2} \sin 2\theta \\ -i \sin \frac{\Delta}{2} \sin 2\theta & \cos \frac{\Delta}{2} + i \sin \frac{\Delta}{2} \cos 2\theta \end{bmatrix} \begin{bmatrix} i & 1 \\ -i & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A_x.e^{i\phi_x} \\ A_y.e^{i\phi_y} \end{bmatrix} e^{i\omega t} \tag{27}
\]

\[ I \propto \sin^2 \frac{\Delta}{2} \tag{12} \]

**Circular polariscope with analyzer at an angle \( \gamma \) with the polariscope X axis:**

\[
E_{\text{final}} = A(\gamma).Q_2.M(\Delta, \theta).Q_2.P(90°) = \\
\begin{bmatrix} \cos^2 \gamma & \cos \gamma \sin \gamma \\ \cos \gamma \sin \gamma & \sin^2 \gamma \end{bmatrix} \begin{bmatrix} i & 1 \\ -i & 1 \end{bmatrix} \begin{bmatrix} \cos \frac{\Delta}{2} - i \sin \frac{\Delta}{2} \cos 2\theta & -i \sin \frac{\Delta}{2} \sin 2\theta \\ -i \sin \frac{\Delta}{2} \sin 2\theta & \cos \frac{\Delta}{2} + i \sin \frac{\Delta}{2} \cos 2\theta \end{bmatrix} \begin{bmatrix} i & 1 \\ -i & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A_x.e^{i\phi_x} \\ A_y.e^{i\phi_y} \end{bmatrix} e^{i\omega t} \tag{28}
\]

\[ I \propto 1 - \cos 2\gamma \cos \Delta - \cos 2\alpha \sin 2\gamma \sin \Delta \tag{15} \]

12. **White Light Photoelasticity**
In the present text photoelasticity has been explained by using only one wave length. Photoelasticity images using monochromatic light characterized by zero light intensity fringes (I=0), full intensity fringes (I=1) and "grey" levels (I=sin²Δ/2), where Δ is the wave length of the emitted light.

If more than one wave length is used, light intensity will be a sum of squared sine functions and the zero intensity for one wave length will not coincide with the others. If white light is used, all wave lengths of the visible light spectrum will be active. When one of these wave lengths is canceled by the photoelastic effect, a particular color (complimentary color) will show in the observed image. In other words, the photoelastic isochromatic response will be composed of different colors that will depend on the retardation caused by the different stress states in the whole field image.

Figure 14 shows colored fringes in a beam under pure bending. They are caused by the cancellation of some wave lengths of the transmitted light. This behavior is a consequence of the different stress states and therefore different retardations present at the loaded beam. Colors resulting from retardation and elimination of certain wave lengths are presented in Figure 14 and its associated Table.

<table>
<thead>
<tr>
<th>Resulting colour</th>
<th>Linear retardation, a (mm)</th>
<th>Fringe order, N</th>
</tr>
</thead>
<tbody>
<tr>
<td>Black</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Grey</td>
<td>180</td>
<td>0.28</td>
</tr>
<tr>
<td>White</td>
<td>260</td>
<td>0.45</td>
</tr>
<tr>
<td>Yellow</td>
<td>345</td>
<td>0.60</td>
</tr>
<tr>
<td>Orange</td>
<td>430</td>
<td>0.79</td>
</tr>
<tr>
<td>Red</td>
<td>620</td>
<td>0.99</td>
</tr>
<tr>
<td>Purple</td>
<td>715</td>
<td>1.19</td>
</tr>
<tr>
<td>Light</td>
<td>620</td>
<td>1.08</td>
</tr>
<tr>
<td>Orange</td>
<td>835</td>
<td>1.33</td>
</tr>
<tr>
<td>Pink</td>
<td>1035</td>
<td>1.82</td>
</tr>
<tr>
<td>Violet</td>
<td>1150</td>
<td>2.00</td>
</tr>
<tr>
<td>Green</td>
<td>1300</td>
<td>2.36</td>
</tr>
</tbody>
</table>

Figure 14: Colors, fringe orders and retardation in white light photoelasticity

14. RGB Photoelasticity

Digital photoelasticity has been advanced since the end of the 70’ies due to advances in computer software and hardware and the introduction of low cost digital image cameras. Several new techniques have been proposed to automate the collection and analysis of isochromatic and isoclinic fringe data, such as the half-fringe photoelasticity, the grey photoelasticity and the red-green-blue (RGB) photoelasticity.

For example, in the RGB photoelasticity technique the light intensity related to each of the three basic colors is decomposed with a $2^8$ (256 intensity levels) resolution. Each level of retardation is represented by a different color that results from an unique reading of the levels of the R, G and B readings. A master table can be built by a calibration experiment as shown in Figure 15. The linear stress and fringe order distributions of the beam are plotted against distance measured in the
photoelastic image. The distance is measured in terms of each pixel location given by the image software. The master table is used to identify the fringe order of an observed point of the loaded model.

Figure 15: Construction of a RGB master table to relate (R,G,B) readings with isochromatic fringe orders. The pure bending calibration beam (left upper corner of the Figure) is made from the same material being tested (model with bolts) and both are examined in the same polariscope and field of view to minimize reading errors.

The isochromatic fringe identification process is based in finding the minimum error value of a R-G-B component matching equation, for example equation (29).

\[ Error^2_i = (R_m - R_{p(i)})^2 + (G_m - G_{p(i)})^2 + (B_m - B_{p(i)})^2 \]  

(29)
Figure 16 summarizes the RGB technique. The color image from a loaded model is digitalized and decomposed in terms of three basic colors or wave lengths \((R_m, G_m, B_m)\). Fringe order of the observed point in the model is determined by finding the \(N_i\) value in the master table which triplet \((R_{p(i)}, G_{p(i)}, B_{p(i)})\) that minimizes equation (29). All sort of procedures can be used together with equation (29) and R-G-B measurements to minimize errors in fringe determination, for example, the fuzzy logic and filtering techniques.

<table>
<thead>
<tr>
<th>I</th>
<th>N</th>
<th>(R_p)</th>
<th>(G_p)</th>
<th>(B_p)</th>
<th>(E^2)</th>
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<tbody>
<tr>
<td>I</td>
<td>I</td>
<td>120</td>
<td>1071</td>
<td>56</td>
<td>146</td>
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<td>122</td>
<td>1100</td>
<td>53</td>
<td>169</td>
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</tr>
<tr>
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<td>123</td>
<td>1114</td>
<td>55</td>
<td>177</td>
<td>175</td>
</tr>
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<td>(.)</td>
<td>(.)</td>
<td>(.)</td>
<td>(.)</td>
<td>(.)</td>
<td>(.)</td>
</tr>
</tbody>
</table>

Figure 16: Reading and identifying the isochromatic fringe order using RGB photoelasticity

15. Determination of the Material Fringe Value

The material fringe \(f_\sigma\) value is determined from calibration by using well known stress states and by reading the corresponding isochromatic fringe orders in the polariscope. When possible, plots of the applied load \(P\), bending moment \(M\) or directly the principal stress difference of an observed point with the measured values of \(N\) should be made to check for linearity and for least square fitting purpose. Figure 17 and equations (30-33) show models commonly used in the material fringe value determination. The stress-optic coefficient \(c\) may also determined by equation (33) using the material fringe value \(f_\sigma\) and the wave length \(\lambda\) of the light used in its determination.
Figure 17: Models commonly used in the calibration of the photoelastic model materials (a) diametrically compressed disk - fringe value \( N \) measured at the center of the disk; (b) beam under pure bending - fringe value measured at the beam surface points, although a plot can be made considering the linear variation of stress and \( N \) with height from the neutral fiber; (c) bar under pure axial loading - fringe value measured at any point due to uniformity of stress and fringe distributions.

\[
\begin{align*}
\text{Diametrically compressed disk:} & \quad f_\sigma = \frac{8P}{\pi D N} \\
\text{Beam under pure bending:} & \quad f_\sigma = \frac{6M}{h^2 N} \\
\text{Bar under tensile loading:} & \quad f_\sigma = \frac{P}{h N} \\
\text{Material fringe value:} & \quad f_\sigma = \frac{1}{\lambda c}
\end{align*}
\]

16. Reflection Photoelasticity

Reflection photoelasticity is an extension of the traditional photoelastic method. It utilizes a thin layer or coating of a birefringent transparent material bonded with a reflective adhesive to the prototype that which will be loaded. The strain state of the surface of the prototype is transmitted to the birefringent coating by the adhesive. Observation of the coating using a reflection polariscope (Figures 18 and 19) provides the full-field strain distribution in the observed area.
The principal strain difference at the observed material point of the prototype is identical to the actual parts under actual loads. It is often used as a method to localize critically stressed areas that will later be analyzed with more localized and accurate techniques.

Some issues to be considered in the design of an experiment using reflection photoelasticity are related to: its accuracy, sensitivity and resolution; the reinforcement effect that it may cause; and the difference in the elastic and thermal properties between coating and prototype materials.

Quantitative reflection photoelasticity is based on the hypothesis of equality of strains in the coating and in the surface points of the prototype (eq. 34). A plane stress state implies that the principal stress normal to the surface is zero.

$$
\begin{align*}
\sigma_z &= 0 \\
\epsilon^c_I &= \epsilon^r_I \\
\epsilon^c_{II} &= \epsilon^r_{II}
\end{align*}
$$ (34)

The principal strain difference at the observed material point of the prototype is identical to the principal strain difference at the coating and is expressed in equation (35) as

$$
\epsilon^c_I - \epsilon^c_{II} = \epsilon^r_I - \epsilon^r_{II} = \frac{N \cdot f_e}{2t}
$$ (35)

where $f_e$ is the strain fringe value of the birefringent coating and the coating thickness is counted twice due to the incident and reflected paths of the light wave. Elastic relations between strains and stresses give:
Figure 19: Reflection photoelasticity analysis of a beam-to-column joint tested to analyze the behavior of the bolted clamped-connection (a) experimental set up; (b) white light photoelastic response of the bolted connection; (c) monochromatic response (yellow light) of the image showed in (b); (d) magnified image of bolted area.

\[
\sigma_i^* - \sigma_{ii}^* = \frac{E^*}{1 + \mu^*} \frac{N.f_\varepsilon}{2t} \tag{36}
\]

\[
\sigma_i^* - \sigma_{ii}^* = \frac{E^*}{E^c} \frac{1 + \mu^c}{1 + \mu^c} \left( \sigma_i^c - \sigma_{ii}^c \right) = \frac{E^*}{E^c} \frac{1 + \mu^c}{1 + \mu^c} \frac{N.f_\sigma}{2t} \tag{37}
\]

The strain fringe value is related to the strain coefficient for the coating K and the light wavelength \(\lambda\) such that:

\[
f_\varepsilon = \frac{\lambda}{K} \tag{38}
\]

17. Stress Separation

The definition of a plane stress state requires three independent pieces of information. The basic photoelastic response furnishes only two pieces: principal stress difference and principal stress direction. Sometimes a third piece of information may be obtained from the type of problem being studied. For example, Figure 20 shows points with uniaxial stress states located at free surfaces. In
general, the heavily loaded points in 2-D stress analysis belong to the free surface points and then methods of stress separation are not needed.

Several methods of stress separation are available in the literature. Some of them are based on hybrid analysis where equilibrium or compatibility equations are employed as additional conditions. Other methods use combinations with experimental (electric analogy, electrical resistance strain gages, special thickness measurements) or numerical solutions (boundary methods, finite elements and finite differences).

A third piece of information can also be obtained from the photoelastic method if the point of interest can be observed by an oblique incident light ray as depicted in Figure 21. The equations (39) and (40) respectively can be written for the normal and oblique incidence observations and therefore stresses can be separated as shown by the resultant equation (41).

Figure 20: Loaded component and boundary conditions (analyzed areas of the component that belong to the free surface) that help stress separation.

Figure 21: Normal and oblique incidence schemes.
\[ \sigma_i - \sigma_{ii} = \frac{N_n}{t} f \]  
(39)

\[ \sigma_i - \sigma_{ii}, \cos^2 \theta = \frac{N_\theta}{t \cos \theta} f_{\sigma} \]  
(40)

\[ \begin{cases} 
\sigma_i = f_{\sigma} \frac{1}{t \sin^2 \theta} (N_\theta - N_n \cos \theta) \\
\sigma_{ii} = f_{\sigma} \frac{1}{t \sin^2 \theta} (N_\theta \cos \theta - N_n) 
\end{cases} \]  
(41)

18. Similitude in 2-D Photoelasticity

Stresses in the prototype material points can be determined by measurements in the models if appropriate similitude laws are applied. In the case of linear and elastic behavior relations between model and prototype stresses are given by equations (42). Load, in-plane and out-of-plane scales are represented in these equations. Variables important to describe the 2-D static linear and elastic problem are schematically presented in Figure 22. It should be noted that fringe distributions can be seen as non dimensional distributions and this fact may be useful in the analysis of whole field stress distributions.

\[ \left( \frac{\sigma^2}{P} \right)_m = \left( \frac{\sigma^2}{P} \right)_p \]

\[ \sigma_P = \sigma_m \frac{P_p}{P_m} \left( \frac{l_m}{l_p} \right)^2 \]

\[ \sigma_P = \sigma_m \frac{P_p}{P_m} \frac{l_m}{l_p} \frac{t_m}{t_p} \]  
(42)

Figure 22: Dimensional variables for model prototype similitude laws

19. Photoelastic Materials

Availability of the best fitted material is one of the most important requirements for success in a photoelastic analysis. All birefringent materials present advantages and draw backs. Certain properties are required in for the analysis. They listed below:

- Good transparency to the wave-length used
- High sensitivity, i.e. small stress difference causes a reasonable number of isochromatic fringes. Mechanical properties must be elastic and linear in the working range
• Optical properties ($f_σ$ and $f_ε$) have to be linear with stress difference
• Homogeneity and isotropy
• Low viscoelastic and low creep responses
• High modulus of elasticity $E$
• High figure of merit $Q = E/f_σ$
• Low influence of time, humidity and machining at or near the free surface of the components
• Availability of sheets to construct 2-D models, and blocks to be machined for 3-D studies, or availability of a liquid to be used in castings of complex shaped 3-D models
• Compatibility with de-molding materials
• Possibility of castings in large volumes without generation of heat
• Low cost.

A list of materials can be found in the literature. Some of them are:

2-D analysis (brand names): Columbia resin CR-39, Homalite 100, PSM-1
2-D analysis (generic): polycarbonate, bakelite, polyester, epoxy resin, glass, gelatin
3-D analysis: epoxy resins with hardener.

<table>
<thead>
<tr>
<th>Table: properties of some photoelastic materials</th>
</tr>
</thead>
<tbody>
<tr>
<td>Material</td>
</tr>
<tr>
<td>Homalite 100 (polyester)</td>
</tr>
<tr>
<td>PS-1 (Measurements Group)</td>
</tr>
<tr>
<td>PSM-1 (Measurements Group)</td>
</tr>
<tr>
<td>Polycarbonate</td>
</tr>
<tr>
<td>Epoxy (100ppw of Araldite F + 40ppw phthalic anhydride; above critical temperature ~ 100°C)</td>
</tr>
</tbody>
</table>

20. Three Dimensional Photoelasticity

The use of the photoelasticity in three dimensional analysis requires some special features due to the complexity caused by the integration of effects suffered by the light wave passing through a varying stress distribution across the thickness of a model. Methods such as the integrated photoelasticity, the scattered light photoelasticity, the tomographic photoelasticity and stress freezing and slice cutting photoelasticity have been used in the past.

The stress freezing and slicing technique has been the most used. It employs polymers such as castings of epoxy resins cured with phytic acid anhydride. The method is schematically presented in Figures 23-24. At room temperature the cured resin presents primary and secondary carbon chains. Heating the resin above the transition or critical temperature breaks the secondary chains. This process lowers the material Young modulus to about 1/100 of its room temperature value. Loading of the model creates strains inversely proportional to the low $E$ modulus. Temperature is allowed to decrease to the room temperature so that the secondary chains can be re-built. This step locks the strains that were produced.
at the higher temperature. Unloading of the model decreases the model strain state of a few percent of the existing strains, proportionally to the E modulus difference at both analyzed temperature. After this strain freezing step the model can be sliced. Each slice is 2-D analyzed as shown in Figure 24. It has to be noted that the strain state is locked by the micro-arrangement of the molecular structure and therefore the frozen state of strain will not be lowered by the slicing process, unless the cutting process influences the temperature distribution and at the zone of cutting the temperatures reach $T_c$ or above.

Figure 23: Sketch of the three-dimensional behavior of a model material during the stress freezing technique. The Young modulus $E$ and therefore the stiffness $K$ of the cured epoxy resin used in the stress freezing technique depend on temperature.

Figure 24 shows a model in its undeformed state when (room) temperature is $T_R$. The model material is loaded by some small loading $P$. The deformed state is generically calculated as $\Delta R = P/K_R$. Temperature is increased to above the critical temperature $T > T_C$. Stiffness decreases dramatically to $K_C$ and then the deformation state becomes $\Delta C = P/K_C$ which is much larger than $\Delta R$. Temperature is then slowly decreased to $T_R$. The deformation state keeps the same but the stiffness of the material molecular structure recuperates the secondary molecule chains. These freeze the primary chains in their highly deformed state, the material stiffness, returning to its high value $K_R$. Loading $P$ is taken off. The deformation state decreases $\Delta R = P/K_R$ but keeps frozen the deformation $\Delta C = P/K_C$.

Figure 24: Sketch of the stress freezing process based on the temperature dependence of the existence of secondary molecule chains (existence $\iff T < T_R$)

The stress states observed in each slice depend on the direction of incidence of the light wave. Figure 25 illustrates this comment by showing incident of light parallel to direction $X$. The stresses that will
cause relative retardation are those orthogonal to the axis of propagation. Consequently, the stress difference that is determined by measuring the fringe order due to light propagating in the X direction, N_X, is caused by the normal stresses \( \sigma_Y \), \( \sigma_Z \), and \( \tau_{YZ} \), as given by equation (43), where t_X is the thickness of the slice considering the direction of measurement X.

\[
(\sigma_I - \sigma_H)_X = \frac{1}{2} \sqrt{(\sigma_Z - \sigma_Y)^2 + 4.\tau^2_{XY}} = \frac{N_X}{t_X} \cdot f_\sigma
\]  

(43)

Figure 25: Photoelastic analysis of a 3-D stress state when the observation light wave is parallel to a given direction

Figure 26: Example of slices cut from a shaft with a U notch loaded under combined bending and torsion
Figure 27: Slice cut from a model that simulates a welded joint of orthogonal tubular members of an off-shore jig. The joint contains a simulated crack.

Bibliography


